7 Solutions of Linear Algebraic Equations

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■ Keywords

- **Algebraic equations** 代数方程组
- **Stiffness matrix** 刚度矩阵
- **Decomposition** 分解
- **Row** 行
- **Column** 列
- **Upper triangular matrix** 上三角矩阵
- **Lower triangular matrix** 下三角矩阵
- **Gaussian elimination** 高斯消元
- **Forward elimination** 向前消元
- **Back substitution** 向后回代

7 Solutions of Linear Algebraic Equations

7.1 LU decomposition method

- By using the finite element method, a standard discrete system is obtained. The basic stiffness equations of the discrete system are linear algebraic equations.

$$\mathbf{K}\hat{\mathbf{u}} = \mathbf{f}$$

- In general, as the number of the linear algebraic equations increasing, it is difficult to solve them manually and directly through constructing the inverse matrix of the stiffness matrix.

$$\hat{\mathbf{u}} = \mathbf{K}^{-1}\mathbf{f}$$

- This section will introduce a commonly used LU decomposition method or triangular decomposition method, also known as Gaussian elimination method, including the forward elimination and back substitution procedures.

7.1 LU decomposition method

- Consider a set of linear algebraic equations, representing the global stiffness equations in finite element method, given by

$$\mathbf{K}\hat{\mathbf{u}} = \mathbf{f}$$

where $\mathbf{K}$ is a $n\times n$ square matrix with known values, representing the global stiffness matrix; $\hat{\mathbf{u}}$ is a vector of unknown parameters, representing the global displacement vector; $\mathbf{f}$ is a vector with known values, representing the global load vector.

7.1 LU decomposition method

- The matrix $\mathbf{K}$ is symmetric and positive definite, which can be written as the product of a lower triangular matrix $\mathbf{L}$ with unit diagonals and an upper triangular matrix $\mathbf{U}$ as follows

$$\mathbf{K} = \mathbf{L}\mathbf{U}$$

LU decomposition, triangular decomposition, Gaussian elimination

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ L_{21} & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & 1 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$$

7.1 LU decomposition method

- The solution to the equations can now be obtained by sequentially solving the pair of equations.

$\mathbf{K} = \mathbf{LU}$

$$\mathbf{K}\hat{\mathbf{u}} = \mathbf{f} \Rightarrow \mathbf{Ly} = \mathbf{f} + \mathbf{U}\hat{\mathbf{u}} = \mathbf{y}$$

- In terms of the individual equations the solution is given by

$\mathbf{Ly} = \mathbf{f}$ — **Forward elimination**

$$y_i = f_i, \quad i = 1$$

$$y_i = f_i - \sum_{j=1}^{i-1} L_{ij} y_j, \quad i = 2,3,\ldots,n$$

$\mathbf{U}\hat{\mathbf{u}} = \mathbf{y}$ — **Back substitution**

$$\hat{u}_i = \frac{y_i}{U_{ii}}, \quad i = n$$

$$\hat{u}_i = \frac{1}{U_{ii}}\left(y_i - \sum_{j=i+1}^{n} U_{ij}\hat{u}_j\right), \quad i = n-1, n-2, \ldots, 1$$

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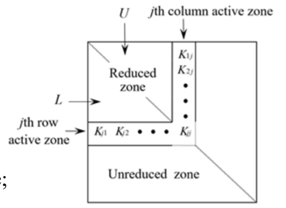
7.1 LU decomposition method

- Based on the organization of Fig. 7.1, it is convenient to consider the coefficient array of stiffness matrix **K** to be divided into three parts

(1) **Reduced zone:** the region which is fully reduced;

(2) **Active zone:** the region which is currently being reduced; where the *j*th column above the diagonal and the *j*th row to the left of the diagonal constitute the active zone;

(3) **Unreduced zone:** the region which contains the original unreduced coefficients.



- **Figure 7.1 Reduced, active, and unreduced zones in LU decomposition of stiffness matrix K.**

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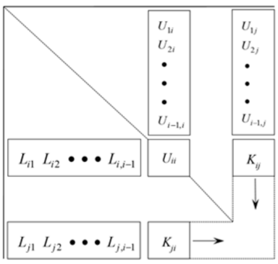
7.1 LU decomposition method

- The coefficients in the decomposed matrices **L** and **U** can be stored in the active zone, respectively, as shown in Fig. 7.2. Through the above processes, the stiffness matrix is decomposed into two matrix parts.



- **Figure 7.2 Matrices L and U stored in active zone.**

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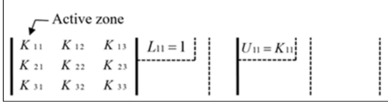
7.1 LU decomposition method

- The algorithm for the LU decomposition of an $n\times n$ square matrix can be deduced from Fig. 7.3 as follows

Step 1. Active zone. First row and column to principal diagonal.



- **Figure 7.3 LU decomposition of stiffness matrix K.**

- First row and colum is activated as

$$U_{11} = K_{11}, \ L_{11} = 1$$

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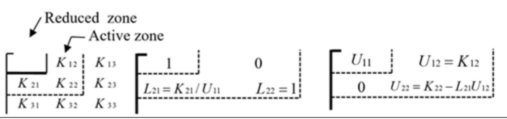
7.1 LU decomposition method

- The algorithm for the LU decomposition of an $n\times n$ square matrix can be deduced from Figs 7.3 as follows

Step 2. Active zone. Second row and column to principal diagonal. Use first row of **K** to eliminate $L_{21}U_{11}$: The active zone uses only values of **K** from the active zone and values of **L** and **U** which have already been computed in steps 1 and 2.



- **Figure 7.3 LU decomposition of stiffness matrix K.**

- For each active zone *j* from 2 to *n*

$$L_{j1} = \frac{K_{j1}}{U_{11}}, \quad U_{1j} = K_{1j} \qquad L_{ji} = \frac{1}{U_{ii}}\left(K_{ji} - \sum_{m=1}^{i-1} L_{jm}U_{mi}\right)$$

$$U_{ij} = K_{ij} - \sum_{m=1}^{i-1} L_{im}U_{mj} \quad i = 2,3,\ldots,j-1$$

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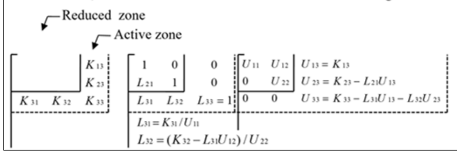
7.1 LU decomposition method

- The algorithm for the LU decomposition of an $n\times n$ square matrix can be deduced from Figs 7.3 as follows

Step 3. Active zone. Third row and column to principal diagonal. Use first row to eliminate $L_{31}U_{11}$: Use second row of reduced terms to eliminate $L_{32}U_{22}$ (reduced coefficient K_{32}).Reduce column 3 to reflect eliminations below diagonal.



- **Figure 7.3 LU decomposition of stiffness matrix K.**

- For each active zone *j* from 2 to *n*

$$L_{j1} = \frac{K_{j1}}{U_{11}}, \quad U_{1j} = K_{1j} \qquad L_{ji} = \frac{1}{U_{ii}}\left(K_{ji} - \sum_{m=1}^{i-1} L_{jm}U_{mi}\right)$$

$$U_{ij} = K_{ij} - \sum_{m=1}^{i-1} L_{im}U_{mj} \quad i = 2,3,\ldots,j-1$$

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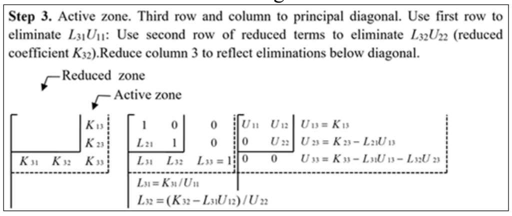
7.1 LU decomposition method

- The algorithm for the LU decomposition of an $n\times n$ square matrix can be deduced from Figs 7.3 as follows



- **Figure 7.3 LU decomposition of stiffness matrix K.**

- finally with L_{jj}=1

$$U_{jj}=K_{jj}-\sum_{m=1}^{j-1}L_{jm}U_{mj}$$

$$L_{ji}=\frac{1}{U_{ii}}\left(K_{ji}-\sum_{m=1}^{i-1}L_{jm}U_{mi}\right)$$

$$U_{ij}=K_{ij}-\sum_{m=1}^{i-1}L_{im}U_{mj}\quad i=2,3,\ldots,j-1$$

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7.1 LU decomposition method

- **Example 7.1** Based on the triangular decomposition of stiffness matrix **K**, compute the solutions of displacements in stiffness equations

$$\mathbf{K}\hat{\mathbf{u}}=\mathbf{f}\qquad \text{with}\quad \mathbf{K}=\begin{bmatrix}4&2&1\\2&4&2\\1&2&4\end{bmatrix},\qquad \mathbf{f}=\begin{Bmatrix}4\\2\\1\end{Bmatrix}$$

- Firstly, the 3×3 stiffness matrix **K** is decomposed, as shown in Table 7.1.

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7.1 LU decomposition method

- **Table 7.1 LU decomposition of 3×3 stiffness matrix K.**

$$U_{11}=K_{11},\ L_{11}=1$$

$$L_{j1}=\frac{K_{j1}}{U_{11}},\quad U_{1j}=K_{1j}$$

$$L_{ji}=\frac{1}{U_{ii}}\left(K_{ji}-\sum_{m=1}^{i-1}L_{jm}U_{mi}\right)$$

$$U_{ij}=K_{ij}-\sum_{m=1}^{i-1}L_{im}U_{mj}\quad i=2,3,\ldots,j-1$$

$$U_{jj}=K_{jj}-\sum_{m=1}^{j-1}L_{jm}U_{mj}$$

K	L	U
Step1. $L_{11}=1$; $U_{11}=4$		
$\begin{bmatrix}4&2&1\\2&4&2\\1&2&4\end{bmatrix}$	$\begin{bmatrix}1&&\\&&\\&&\end{bmatrix}$	$\begin{bmatrix}4&&\\&&\\&&\end{bmatrix}$
Step2. $L_{21}=\frac{2}{4}=0.5$; $U_{12}=2$; $U_{22}=4-0.5\times2=3$		
$\begin{bmatrix}&2&1\\2&4&2\\1&2&4\end{bmatrix}$	$\begin{bmatrix}1&&\\0.5&1&\\&&\end{bmatrix}$	$\begin{bmatrix}4&2&\\&3&\\&&\end{bmatrix}$
Step3. $L_{31}=\frac{1}{4}=0.25$; $U_{13}=1$; $L_{32}=\frac{2-0.25\times2}{3}=\frac{1.5}{3}=0.5$; $U_{23}=2-0.5\times1=1.5$; $L_{33}=1$; $U_{33}=4-0.25\times1-0.5\times1.5=3$		
$\begin{bmatrix}&&1\\&&2\\1&2&4\end{bmatrix}$	$\begin{bmatrix}1&&\\0.5&1&\\0.25&0.5&1\end{bmatrix}$	$\begin{bmatrix}4&2&1\\&3&1.5\\&&3\end{bmatrix}$
Step4. Check		
$\begin{bmatrix}1&&\\0.5&1&\\0.25&0.5&1\end{bmatrix}\begin{bmatrix}4&2&1\\&3&1.5\\&&3\end{bmatrix}=\begin{bmatrix}4&2&1\\2&4&2\\1&2&4\end{bmatrix}$		

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7.1 LU decomposition method

- Secondly, using the decomposed matrices **L** and **U**, the forward elimination and back substitution procedures will be implemented.

Forward elimination

$$\begin{bmatrix}1&&\\0.5&1&\\0.25&0.5&1\end{bmatrix}\begin{bmatrix}4&2&1\\&3&1.5\\&&3\end{bmatrix}$$

$$y_i=f_i,\quad i=1$$

$$y_i=f_i-\sum_{j=1}^{i-1}L_{ij}y_j,\quad i=2,3,\ldots,n$$

$$i=1,\quad y_i=f_i=f_1=4$$

$$i=2,\quad y_i=f_i-\sum_{j=1}^{i-1}L_{ij}y_j=f_2-\sum_{j=1}^{1}L_{21}y_1=2-0.5\times4=0$$

$$i=3,\quad y_i=f_i-\sum_{j=1}^{i-1}L_{ij}y_j=f_3-\sum_{j=1}^{2}L_{3j}y_j=1-0.25\times4-0.5\times0=0$$

$$\mathbf{f}=\begin{Bmatrix}4\\2\\1\end{Bmatrix}$$

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7.1 LU decomposition method

- Secondly, using the decomposed matrices **L** and **U**, the forward elimination and back substitution procedures will be implemented.

Back substitution

$$\begin{bmatrix}1&&\\0.5&1&\\0.25&0.5&1\end{bmatrix}\begin{bmatrix}4&2&1\\&3&1.5\\&&3\end{bmatrix}$$

$$\hat{u}_i=\frac{y_i}{U_{ii}},\quad i=n$$

$$\hat{u}_i=\frac{1}{U_{ii}}\left(y_i-\sum_{j=i+1}^{n}U_{ij}\hat{u}_j\right),\quad i=n-1,n-2,\ldots,1$$

$$i=3,\quad \hat{u}_i=\frac{y_i}{U_{ii}}=\frac{y_3}{U_{33}}=\frac{0}{3}=0$$

$$i=2,\quad \hat{u}_i=\frac{1}{U_{ii}}\left(y_i-\sum_{j=i+1}^{n}U_{ij}\hat{u}_j\right)=\frac{1}{U_{22}}\left(y_2-\sum_{j=3}^{3}U_{23}\hat{u}_3\right)=\frac{1}{3}(0-1.5\times0)=0$$

$$i=1,\quad \hat{u}_i=\frac{1}{U_{ii}}\left(y_i-\sum_{j=i+1}^{n}U_{ij}\hat{u}_j\right)=\frac{1}{U_{11}}\left(y_1-\sum_{j=2}^{3}U_{1j}\hat{u}_j\right)$$

$$=\frac{1}{4}\left(4-U_{12}\hat{u}_2+U_{13}\hat{u}_3\right)=\frac{1}{4}\left(4-2\times0+1\times0\right)=1$$

$$\mathbf{y}=\begin{Bmatrix}4\\0\\0\end{Bmatrix}$$

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7.1 LU decomposition method

- The displacement solutions are obtained

$$\hat{\mathbf{u}}=\{\hat{u}_1\quad \hat{u}_2\quad \hat{u}_3\}^T=\{1\quad 0\quad 0\}^T$$

- According the below verification, these solutions are completely reliable

$$\mathbf{K}\hat{\mathbf{u}}=\begin{bmatrix}4&2&1\\2&4&2\\1&2&4\end{bmatrix}\begin{Bmatrix}1\\0\\0\end{Bmatrix}=\begin{Bmatrix}4\\2\\1\end{Bmatrix}=\mathbf{f}$$

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The End

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